An Introduction To The Theory Of Automorphic Functions

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AN INTRODUCTION
TO THE THEORY OF
AUTOMORPHIC FUNCTIONS

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PREFACE

Owing largely to the researches of Poincaré and Klein the domain of Automorphic Functions has expanded enormously during the last thirty-five years; and the ramifications of the subject have extended into many and diverse fields. This has caused embarrassment in the selection of materials for a book of modest dimensions, and has necessitated a brief treatment, or in some cases the exclusion, of many important and attractive subjects. The aim throughout has been to present in as thorough a manner as possible the concepts and theorems on which the theory is founded, and to describe in less detail certain of its important developments.

The present tract had its origin in a series of lectures on Automorphic Functions given to the Mathematical Research Class of the University of Edinburgh during the Spring Term of 1915.

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L. R. F.

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CHAPTER 1

LINEAR TRANSFORMATIONS

1. The Linear Transformation.—If $z$ is a complex quantity whose real part is $x$ and whose imaginary part is $iy$, it is customary to represent $z$ by a point in a plane whose abscissa is $x$ and whose ordinate is $y$, the coordinates being referred to perpendicular axes. If $x$, or $y$, or both become infinite, the point recedes to an infinite distance from the origin of coordinates. This is expressed by $z = \infty$, and infinity is considered to be a single point in the complex plane. In this respect the conception of infinity differs from that employed in projective geometry, where the infinite region is considered to be a line.*

Consider $z' = f(z)$, where $f(z)$ is a function of $z$, and let the variable $z'$ be represented on a second plane. To each point $z$ for which the function $f(z)$ is defined there correspond one or more values of $z'$ given by the above equation. To points, curves, and areas in the $z$-plane there correspond by virtue of the equation points, curves, and areas in the $z'$-plane.

The configurations in the one plane are said to be transformed into configurations in the other plane. The whole theory of automorphic functions depends upon a particular type of transformation, defined as follows:—

Definition.—The transformation

$$z' = \frac{az + b}{cz + d} \quad \ldots \quad (1)$$

where $a$, $b$, $c$, $d$ are constants (real or complex) and $ad - bc \neq 0$ is called a linear transformation.†

* The reason for this is that the kind of transformations most frequently considered in the theory of functions of a complex variable transform the infinite region into a point in the finite part of the plane; whereas ordinary projection in geometry transforms the infinite region into a line.

† Also called a linear substitution, a homographic transformation, or a homographic substitution. If $ad - bc = 0$, the equation reduces to $z' = \text{constant}$; but this case is without interest.
The quantity \( ad - bc \) is called the determinant of the transformation. It will be convenient to have always

\[
ad - bc = 1 .
\]

(2)

the determinant in the general case taking this value if the numerator and denominator of the fraction in the second member be divided by \( \pm \sqrt{(ad - bc)} \).

Let equation (1) be solved for \( z \).

\[
z = \frac{-az' + b}{cz' - a} .
\]

(3)

This is called the inverse of the transformation (1). It is a linear transformation with the same determinant as (1). We state then—

THEOREM 1.—The inverse of a linear transformation is a linear transformation.

To each value of \( z \) there corresponds one, and only one, value of \( z' \) by (1); and conversely to each value of \( z' \) there corresponds one, and only one, value of \( z \) by (3). In other words, the \( z \)-plane is transformed into the \( z' \)-plane in a one-to-one manner.

It will be most serviceable to represent the values of \( z' \) not on a different plane, but on the same plane and with the same system of coordinates as are used for representing \( z \). In all that follows but one plane will be used unless the contrary is explicitly stated. The results just found can be stated in the following—

THEOREM 2.—The \( z \)-plane is transformed into itself in a one-to-one manner by a linear transformation.

Let the points \( z' \) be subjected to a linear transformation,

\[
z'' = \frac{az' + \beta}{cz' + \delta} .
\]

(4)

Expressing \( z'' \) as a function of \( z \),

\[
z'' = \frac{a[(az + b)/(cz + d)] + \beta}{\gamma[(az + b)/(cz + d)] + \delta} = \frac{(aa + \beta c)z + ab + \beta d}{(\gamma a + \delta c)z + \gamma b + \delta d} .
\]

(5)

Making the transformation (1) and then making (4) is equivalent to the single transformation (5). Now (5) is also a linear transformation; its determinant in the form in which the fraction is written is \( (ad - bc)(a\delta - \beta \gamma) \). It is worth noting that if the determinants of (1) and (4) are each unity, the determinant of
(5) is unity also without further change. If \( z'' \) be subjected to a linear transformation, we get again a linear transformation of \( z \), and so on. Thus we have, in general—

**Theorem 3.**—The successive performance of any number of linear transformations is equivalent to a single linear transformation.

The linear, or homographic, transformation has the property that it leaves invariant the cross-ratio of four points. For if \( z_1, z_2, z_3, z_4 \) are four points, and \( z'_1, z'_2, z'_3, z'_4 \) are the points into which they are transformed, we have, on substituting for \( z'_1, z'_2, z'_3, z'_4 \) their values from (1),

\[
\frac{(z'_1 - z'_2)(z'_3 - z'_4)}{(z'_1 - z'_3)(z'_2 - z'_4)} = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)} .
\]  

(6)

**2. The Fixed Points of the Transformation.**—The points which remain unchanged under the transformation (1) are found by solving the equation

\[
z = (az + b)/(cz + d), \quad \text{or} \quad cz^2 + (d - a)z - b = 0 .
\]  

(7)

If \( c + 0 \), the fixed points are

\[
\xi_1 = \frac{a - d + \sqrt{(a - d)^2 + 4bc}}{2c} ; \quad \xi_2 = \frac{a - d - \sqrt{(a - d)^2 + 4bc}}{2c} .
\]  

(8)

If \( c = 0 \), the fixed points are

\[
\xi_1 = \infty ; \quad \xi_2 = b/(d - a).
\]  

(9)

There is but one fixed point if \((a - d)^2 + 4bc = 0\); or, when \(ad - bc = 1\), if \((a + d)^2 = 4\). It is the point

\[
\xi = (a - d)/2c .
\]  

(10)

Equation (7) will have more than two roots only in case it is identically satisfied; that is, if \( c = d - a = b = 0 \). In this case every point is unchanged in position.

**Theorem 4.**—The only linear transformation with more than two fixed points is the identical transformation \( z' = z \).

By means of the last theorem we are able to establish the following proposition—

**Theorem 5.**—There is one, and only one, linear transformation which transforms three distinct points \( z_1, z_2, z_3 \) into three distinct points \( z'_1, z'_2, z'_3 \).

We shall prove first that there is not more than one such trans-
formation. Let \( z' = (ax + b)(cz + d) \) and \( z'' = (az + \beta)(cz + \delta) \) be two transformations carrying \( z_1, z_2, z_3 \) into \( z'_1, z'_2, z'_3 \). Eliminating \( z \), we get \( z'' \) as a linear function of \( z' \); \( z'' = (Az' + B)/(Cz' + D) \). We know from Theorems 1 and 3, without actually performing the elimination, that \( z'' \) depends linearly upon \( z' \); for \( z \) is derived from \( z' \), and \( z'' \) from \( z \), by a linear transformation. When \( z = z_1, z_2, \) or \( z_3 \), we have by hypothesis \( z'' = z'_1, z'_2, \) or \( z'_3 \). The transformation from \( z' \) to \( z'' \) has then three fixed points, \( z'_1, z'_2, z'_3 \); and by Theorem 4 \( z'' = z' \). The two transformations are the same, which proves that there is not more than one transformation with the desired properties.

That there is always one such transformation we prove by actually setting it up. If none of the six values is infinite, consider the transformation defined by

\[
\frac{z' - z'_1}{z' - z'_2} \frac{z'_3 - z'_2}{z'_3 - z'_1} = \frac{z - z_1}{z - z_2} \frac{z_3 - z_2}{z_3 - z_1}.
\]

(11)

an equation which expresses the equality of the cross-ratios \((z'z'_1, z'_2z'_3)\) and \((zz_1, z_2z_3)\). This is of the form (1) when solved for \( z' \) in terms of \( z \). It obviously transforms \( z_1, z_2, z_3 \) into \( z'_1, z'_2, z'_3 \). The equation (11) depends upon both variables if each set of three points is distinct; the determinant of the transformation then does not vanish [see footnote, Section 1].

If \( z_1 = \infty, z_2 = \infty, \) or \( z_3 = \infty \), it is necessary to replace the second member of (11) by

\[
\frac{z_2 - z_3}{z - z_2}, \quad \frac{z - z_1}{z_3 - z_1}, \quad \text{or} \quad \frac{z - z_1}{z - z_2}
\]

respectively; and a similar change is necessary in the first member for an infinite value of \( z'_1, z'_2, \) or \( z'_3 \). In any case there is one transformation with the desired properties, and the theorem is established.

Equation (11) is a convenient form for use in actually setting up the transformation carrying three given points into three given points.

3. Conformal Transformations.—Let \( p \) be a point in a plane, and let \( C_1 \) and \( C_2 \) be two curves issuing from \( p \). Let \( \theta \) be the angle between the curves (i.e., the angle between their tangents at \( p \)). For convenience we shall consider the angle positive if in passing from a point on \( C_1 \) to a point on \( C_2 \) through the region