Elements of Quaternions, Volume 2

Hamilton William Rowan
ELEMENTS OF QUATERNIONS.
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QUATERNIONS.

BY THE LATE
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SECOND EDITION.

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VOLUME II.

LONGMANS, GREEN, AND CO.,
39, PATERNOSTER ROW, LONDON,
NEW YORK, AND BOMBAY.
1901.
ADVERTISEMENT TO THE SECOND EDITION.

I have reserved for the Appendix to this Volume the longer additional and illustrative notes which I have written for the new edition of the "Elements."

Some of those notes would have been inconveniently long as footnotes; others would have been inconveniently placed. For example, although the Note on Screws relates naturally to Art. 416 and that on the Kinematical Treatment of Curves to Art. 396, I have placed the Note on Screws before the Note on Curves because Hamilton's remarks on screw motion in the earlier Article required some development in order to make the Note on Curves easily intelligible. Accordingly the order of the notes has been arranged with reference to the notes themselves rather than with reference to the text. The selection and treatment of the subjects of these notes have been subordinated to the illustration of quaternion methods. I have not hesitated to sacrifice brevity for suggestiveness, and above all I have tried to render the notation as explicit as possible.

An analysis of the Appendix will be found on pages xlv-xlxi.

For greater convenience I have provided an Index to the whole work referring to the pages, the volumes being distinguished by the numbers i and ii.

I take this opportunity of testifying to the extraordinary accuracy both of matter and of printing in the first edition of the "Elements." Every portion of the work bears evidence of Hamilton's unsparing pains. I cannot recall a single sentence ambiguous in its meaning, or a single case in which a difficulty is not honestly faced. I see no sign of diminished vigour or of relaxed care in those portions of the work written in his failing health. My task as editor has convinced me of the extreme caution with which any endeavour should be made to improve or modify the calculus of Quaternions.

In conclusion, I desire to express my thanks to the College Printer, Mr. George Weldrick, for the great care he has taken in printing this edition for the Board of Trinity College, and for his unvarying courtesy to myself.

CHARLES JASPER JOLY.

The Observatory, Dunsink,
10th December, 1900.
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### CHAPTER III.

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In these Sections, $dp$ usually denotes a tangent to a curve, and $v$ a normal to a surface. Some of the theorems or constructions may perhaps be new; for instance, those connected with the *cone of parallels* (pp. 6, 26, &c.) to the tangents to a curve of double curvature; and possibly the theorem (p. 42), respecting *reciprocal curves in space*: at least, the deductions here given of these results may serve as exemplifications of the Calculus employed. In treating of *Families of Surfaces* by quaternions, a sort of *analogue* (pp. 47, 48) to the formation and integration of *Partial Differential Equations* presents itself; as indeed it had done, on a similar occasion, in the *Lectures* (574).

- **Section 6.** On Osculating Circles and Spheres, to Curves in Space; with some connected Constructions, .... 50-179

The analysis, however condensed, of this long Section (III. iii. 6), cannot conveniently be performed otherwise than under the heads of the respective *Articles* (388-401) which compose it: each Article being followed by several sub-articles, which form with it a sort of *Series.*

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* A *Table of initial Pages* of all the *Articles* will be elsewhere given, which will much facilitate reference.
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\[ \text{Vector of Curvature} = (r - κ)^{-1} = \frac{dUdp}{\mathbf{T}dp} = \frac{1}{dp} \mathbf{V} \frac{dp}{dp} = \&c.; \]  

(S)

and if the arc (p) of the curve be made the independent variable, then

\[ \text{Vector of Curvature} = r^2 = D^2p = \frac{d^2p}{ds^2}. \]  

(S')

Examples: curvatures of helix, ellipse, hyperbola, logarithmic spiral; locus of centres of curvature of helix, plane evolute of plane ellipse,

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ARTICLE 396.—Notations τ, τ',… for D, Ds, Ps, &c.; properties of a curve depending on the square (τ²) of its arc, measured from a given point p; τ = unit-tangent, τ' = vector of curvature, τ-1 = T' = curvature (or first curvature, comp. Art. 397), \( \nu = \tau \tau' = \text{bivector} \); the three planes, respectively perpendicular to τ, τ', ν, are the normal plane, the rectifying plane, and the osculating plane; general theory of enantomic lines and planes, vector of rotation, axis of displacement, osculating screw surface; condition of developability of surface of enantomics,

ARTICLE 397.—Properties depending on the cube (τ³) of the arc; Radius r (denoted here, for distinction, by a roman letter), and Vector r-1, of Second Curvature; this radius r may be either positive or negative (whereas the radius r of first curvature is always treated as positive), and its reciprocal r-1 may be thus expressed (pp. 92, 98),

\[ \text{Second Curvature} = r^{-1} = S \frac{d^3p}{Vdpd^2p}, (T), \text{ or, } r^{-1} = S \frac{\tau'}{\tau'}, \]  

(T')

the independent variable being the arc in (T), while it is arbitrary in (T); but quaternions

* In this Article, or Series, 397, and indeed also in 396 and 398, several references are given to a very interesting Memoir by M. de Saint-Venant, "Sur les lignes courbes non planes"; in which, however, that able writer objects to such known phrases as second curvature, torsion, &c., and proposes in their stead a new name "cauchur," which it has not been thought necessary here to adopt. (Journal de l'École Polytechnique, Cahier xxx.)