Introduction to the theory of algebraic equations

Dickson Leonard E
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Author: Dickson Leonard E

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INTRODUCTION TO THE

THEORY OF

ALGEBRAIC EQUATIONS.

BY

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# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>CHAPTER</th>
<th>PAGES</th>
</tr>
</thead>
<tbody>
<tr>
<td>I. Solution of the General Quadratic, Cubic, and Quartic Equations.</td>
<td>1–9</td>
</tr>
<tr>
<td>Lagrange's Theorem on the Irrationalities Entering the Roots.</td>
<td></td>
</tr>
<tr>
<td>Exercises</td>
<td>4</td>
</tr>
<tr>
<td>II. Substitutions; Rational Functions.</td>
<td>10–14</td>
</tr>
<tr>
<td>Exercises</td>
<td>14</td>
</tr>
<tr>
<td>III. Substitution Groups; Rational Functions.</td>
<td>15–26</td>
</tr>
<tr>
<td>Exercises</td>
<td>20</td>
</tr>
<tr>
<td>IV. The General Equation from the Group Standpoint.</td>
<td>27–41</td>
</tr>
<tr>
<td>Exercises</td>
<td>41</td>
</tr>
<tr>
<td>V. Algebraic Introduction to Galois' Theory.</td>
<td>42–47</td>
</tr>
<tr>
<td>VI. The Group of an Equation.</td>
<td>48–63</td>
</tr>
<tr>
<td>Exercises</td>
<td>57–58</td>
</tr>
<tr>
<td>VII. Solution by means of Resolvent Equations.</td>
<td>64–72</td>
</tr>
<tr>
<td>VIII. Regular Cyclic Equations; Abelian Equations.</td>
<td>73–78</td>
</tr>
<tr>
<td>IX. Criterion for Algebraic Solvability</td>
<td>79–86</td>
</tr>
<tr>
<td>X. Metacyclic Equations; Galoisian Equations.</td>
<td>87–93</td>
</tr>
<tr>
<td>XI. An Account of More Technical Results.</td>
<td>94–98</td>
</tr>
</tbody>
</table>

## APPENDIX

- Symmetric Functions. .................................................. 99–101
- On the General Equation ................................................ 101–102

**INDEX** .............................................................................. 103–104
PREFACE.

The solution of the general quadratic equation was known as early as the ninth century; that of the general cubic and quartic equations was discovered in the sixteenth century. During the succeeding two centuries many unsuccessful attempts were made to solve the general equations of the fifth and higher degrees. In 1770 Lagrange analyzed the methods of his predecessors and traced all their results to one principle, that of rational resolvents, and proved that the general quintic equation cannot be solved by rational resolvents. The impossibility of the algebraic solution of the general equation of degree \( n (n > 4) \), whether by rational or irrational resolvents, was then proved by Abel, Wantzel, and Galois. Out of these algebraic investigations grew the theory of substitutions and groups. The first systematic study of substitutions was made by Cauchy (Journal de l'école polytechnique, 1815).

The subject is here presented in the historical order of its development. The First Part (pp. 1–41) is devoted to the Lagrange-Cauchy-Abel theory of general algebraic equations. The Second Part (pp. 42–98) is devoted to Galois' theory of algebraic equations, whether with arbitrary or special coefficients. The aim has been to make the presentation strictly elementary, with practically no dependence upon any branch of mathematics beyond elementary algebra. There occur numerous illustrative examples, as well as sets of elementary exercises.

In the preparation of this book, the author has consulted, in addition to various articles in the journals, the following treatises:

The author takes this opportunity to express his indebtedness to the following lecturers whose courses in group theory he has attended: Oscar Bolza in 1894, E. H. Moore in 1895, Sophus Lie in 1896, Camille Jordan in 1897.

But, of all the sources, the lectures and publications of Professor Bolza have been of the greatest aid to the author. In particular, the examples (§ 65) of the group of an equation have been borrowed with his permission from his lectures.

The present elementary presentation of the theory is the outcome of lectures delivered by the author in 1897 at the University of California, in 1899 at the University of Texas, and twice in 1902 at the University of Chicago.

Chicago, August, 1902
THEORY OF ALGEBRAIC EQUATIONS.

FIRST PART.

THE LAGRANGE-ABEL-CAUCHY THEORY OF GENERAL ALGEBRAIC EQUATIONS.

CHAPTER I.

SOLUTION OF THE GENERAL QUADRATIC, CUBIC, AND QUARTIC EQUATIONS. LAGRANGE'S THEOREM* ON THE IRRATIONALITIES ENTERING THE ROOTS.

1. Quadratic equation. The roots of \( x^2 + px + q = 0 \) are

\[
x_1 = \frac{1}{2} (-p + \sqrt{p^2 - 4q}), \quad x_2 = \frac{1}{2} (-p - \sqrt{p^2 - 4q}).
\]

By addition, subtraction, and multiplication, we get

\[
x_1 + x_2 = -p, \quad x_1 - x_2 = \sqrt{p^2 - 4q}, \quad x_1 x_2 = q.
\]

Hence the irrationality \( \sqrt{p^2 - 4q} \), which occurs in the expressions for the roots, is rationally expressible in terms of the roots, being equal to \( x_1 - x_2 \). Unlike the last function, the functions \( x_1 + x_2 \) and \( x_1 x_2 \) are symmetric in the roots and are rational functions of the coefficients.

2. Cubic equation. The general cubic equation may be written

\[
x^3 - c_1 x^2 + c_2 x - c_3 = 0.
\]

Setting \( x = y + \frac{1}{3} c_1 \), the equation (1) takes the simpler form

\[
y^3 + py + q = 0,
\]

* Réflexions sur la résolution algébrique des équations, Œuvres de Lagrange, Paris, 1869, vol. 3; first printed by the Berlin Academy, 1770-71.
if we make use of the abbreviations

\[ p = c_2 - \frac{1}{3}c_1^2, \quad q = -c_3 + \frac{3}{2}c_1c_2 - \frac{c_3}{2}c_1^3. \]

The cubic (2), lacking the square of the unknown quantity, is called the reduced cubic equation. When it is solved, the roots of (1) are found by the relation \( x = y + \frac{1}{3}c_1 \).

The cubic (2) was first solved by Scipio Ferreto before 1505. The solution was rediscovered by Tartaglia and imparted to Cardan under promises of secrecy. But Cardan broke his promises and published the rules in 1545 in his \textit{Ars Magna}, so that the formulae bear the name of Cardan. The following method of deriving them is essentially that given by Hudde in 1650. By the transformation

\[ y = z - \frac{p}{3z}, \]

the cubic (2) becomes

\[ z^3 - \frac{p^3}{27z^3} + q = 0, \]

whence

\[ z^6 + qz^3 - \frac{p^3}{27} = 0. \]

Solving the latter as a quadratic equation for \( z^3 \), we get

\[ z^3 = -\frac{1}{2}q \pm \sqrt{R}, \quad R = \frac{1}{4}q^2 + \frac{1}{2}p^3. \]

Denote a definite one of the cube roots of \(-\frac{1}{2}q + \sqrt{R}\) by

\[ \sqrt[3]{-\frac{1}{2}q + \sqrt{R}}. \]

The other two cube roots are then

\[ \omega \sqrt[3]{-\frac{1}{2}q + \sqrt{R}}, \quad \omega^2 \sqrt[3]{-\frac{1}{2}q + \sqrt{R}}, \]

where \( \omega \) is an imaginary cube root of unity found as follows. The three cube roots of unity are the roots of the equation

\[ r^3 - 1 = 0, \quad \text{or} \quad (r - 1)(r^2 + r + 1) = 0. \]

The roots of \( r^2 + r + 1 = 0 \) are \( -\frac{1}{2} + \frac{1}{2}\sqrt{-3} = \omega \) and \( -\frac{1}{2} - \frac{1}{2}\sqrt{-3} = \omega^2 \). Then

\[ \omega^3 + \omega + 1 = 0, \quad \omega^3 = 1. \]